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# GEODETIC RESEARCH STUDIES

Final Technical Report

Contract NSR 09-015-054

September 1968

Principal Investigators: E. M. Gaposchkin
Dr. C. A. Lundquist
Dr. G. Veis

Project Administrator : R. W. Martin

Prepared for

National Aeronautics and Space Administration
Washington, D. C. 20546

Smithsonian Institution
Astrophysical Observatory
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# GEODETIC RESEARCH STUDIES

Final Technical Report
Task 03, NSR 09-015-054

# INTERFACE WITH SATELLITE ALTIMETERS

## 1. INTRODUCTION

This study undertook to:

A. Investigate the usefulness of altimeter data as a means of measurement of the geopotential surface of the ocean for the purpose of extending the knowledge of the gravitational field of the earth.

B. Consider how altimeter data can be merged with satellite-tracking data in the accomplishment of this objective.

C. Consider implications of A and B and related research topics for future study of altimeter-data requirements.

These investigations are now completed to the depth provided for by the contract funding.

## 2. RESULTS AND DOCUMENTATION

Conclusions for item B were the first to be reached and reported. Dr. C. A. Lundquist initially presented these orally at the 1967 NASA Electronics Research Center Seminar on Guidance Theory and Trajectory Analysis, May 31 to June 1, 1967 (see Attachment A, The Interface Between Satellite Altimetry and Orbit Determination, abstract). A somewhat expanded discussion of this topic is contained in SAO Special Report No. 248 (Attachment B, Satellite Altimetry and Orbit Determination).

In brief, Special Report No. 248 concludes that ocean-to-satellite altitudes can be treated in the same manner as other satellite-tracking data. The fundamental equations for this treatment are very similar in form to the equations used for station-to-satellite range measurements. The appropriate formulas are in the report. It should be emphasized that this approach involves the simultaneous improvement of the orbit of the satellite and the geopotential representation.

Special Report No. 248 does not consider in depth what geopotential details are derivable from altitude data. This matter is discussed more fully in the first paper of a forthcoming SAO Special Report (Attachment C, Possible Geopotential Improvement from Satellite Altimetry). This paper answers the questions raised by topic A to the extent that was possible within the resources available to this investigation.

Through degree and order 15, 15 in a spherical harmonic expansion for the geopotential, or even to somewhat higher indices, no particular troubles seem likely in following the procedures suggested in Special Report No. 248. At considerably higher degree and order, a problem arises because the number of terms becomes very large. For example, through 36, 36, there are some 1369 terms; that is, the potential at any point is calculated by summing this many terms in a series. Using for the 36, 36 case the approach discussed for the 15, 15 would imply the inversion of a $1369 \times 1369$ matrix. The situation is even more unwieldy for still finer geopotential detail that the altimeter may measure.

This problem might be largely circumvented by use of instead of spherical harmonics a different but equivalent set of functions to represent the geopotential. At any point on the geoid only a very few of these alternative functions have significant values; the rest are insignificant. In relation to topic C, this possibility has been investigated in some detail, as it seems to offer a solution to the principal remaining problem in using altitude data to improve the geopotential.

An introduction to the mathematical concepts involved in these alternative functions was presented orally by Dr. Giacaglia at the 1968 NASA Electronics Research Center Seminar on Guidance Theory and Trajectory Analysis (Attachment D, Representations for Fine Geopotential Structure, abstract). A further elaboration of this mathematical formalism is included in the first paper (Lundquist and Giacaglia) in the forthcoming SAO Special Report (Attachment C). A second paper (Hebb and Mair) in that report presents specific examples of the alternative functions.

Although Special Report No. 248 does include a brief history of satellite altimeters and a statement of the reasons for using such instrumentation, it considers in detail only the application to geopotential refinement. However, the possibilities for routine on-board satellite tracking by altimeter deserve recognition. This fact was emphasized in a working paper (Objectives of Satellite Altimetry) prepared for a NASA Headquarters meeting on June 13, 1968; the working paper drew upon results from the investigations under this contract.

Because other groups were investigating altimeter hardware, only a minimal review of this topic was undertaken. However, one meeting of the SAO 1968 Summer Seminar on Accurate Tracking Techniques and Problems was devoted to altimeter systems. The resulting paper by H. Albers will be included in the seminar proceedings to be published as an SAO Special Report.

Finally, the principal conclusion of the whole investigation is worth repeating here: Ocean-to-satellite altitudes promise substantial geopotential information in a form that can be analyzed without undue difficulty. Thus, the satellite altimeter can probably provide the next major advance in our understanding of the geopotential.

ATTACHMENT A

# GUIDANCE THEORY AND TRAJECTORY ANALYSIS SEMINAR

KRESGE AUDITORIUM
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 31 - June 1, 1967

Sponsored by

NATIONAL AERONAUTICS
AND SPACE ADMINISTRATION
ELECTRONICS RESEARCH CENTER

Cambridge, Massachusetts

# THE INTERFACE BETWEEN SATELLITE ALTIMETRY AND ORBIT DETERMINATION

By Dr. Charles A. Lundquist
Smithsonian Astrophysical Observatory
Cambridge, Massachusetts

## ABSTRACT

The technological problems associated with altimetry from a satellite are the subject of wide investigation in the United States. Therefore, it is reasonable to assume that accurate altitudes measured from satellites will eventually be available, and it is prudent to ask now what are the interfaces between altitude measurements and satellite orbit determination practices.

From one point of view, accurate altitude data may generate accuracy requirements which must be met by orbit determination procedures. From another point of view, the altitudes themselves may be used as tracking data in orbit determination. If the altitude of a satellite above the ocean surface is obtained, this may be viewed as a measured relationship between a point on an equipotential surface of the geopotential and satellite position determined by the equations of motion derived from the geopotential.

These various interfaces can be explored in the context of the procedures used at the Smithsonian Astrophysical Observatory for orbit determination and geophysical research.

## DR. CHARLES J. LUNDQUIST

Dr. Lundquist is Assistant Director (Science) at the Smithsonian Institution Astrophysical Observatory, a position which he has held since 1962. He obtained his Ph.D. degree in Physics from the University of Kansas in 1954. He was Chief of the Physics and Astrophysics Section, Research Projects Laboratory, Redstone Arsenal from 1956-1960. Subsequently he was appointed Director, Supporting Research Office, and Chief, Physics and Astrophysics Branch, Research Projects Division, Marshall Space Flight Center from 1960-1962.

MOTIVES FOR SATELLITE ALTIMETRY

GUIDANCE AND TRAJECTORY ANALYSIS

Orbit analysts say they can determine spacecraft trajectories if oceanography gives them an accurate shape of the ocean surface.

OCEANOGRAPHY

Oceanographers say they can measure the shape of the ocean if orbit analysis gives them accurate satellite positions.

Who is the chicken and which is the egg?

Figure 1.

ESTABLISHED PRACTICE IN SATELLITE GEODESY

A TRACKING MEASUREMENT YIELDS AN OBSERVATION EQUATION relating tracking station coordinates, satellite orbital elements and geophysical parameters.

THE OBSERVATION EQUATIONS ARE USED SIMULTANEOUSLY OR CYCLICLY to refine station coordinates, orbital elements and geophysical parameters.

OBSERVATIONS OF DIFFERENT NATURE CAN BE COMBINED in the refinement calculation.

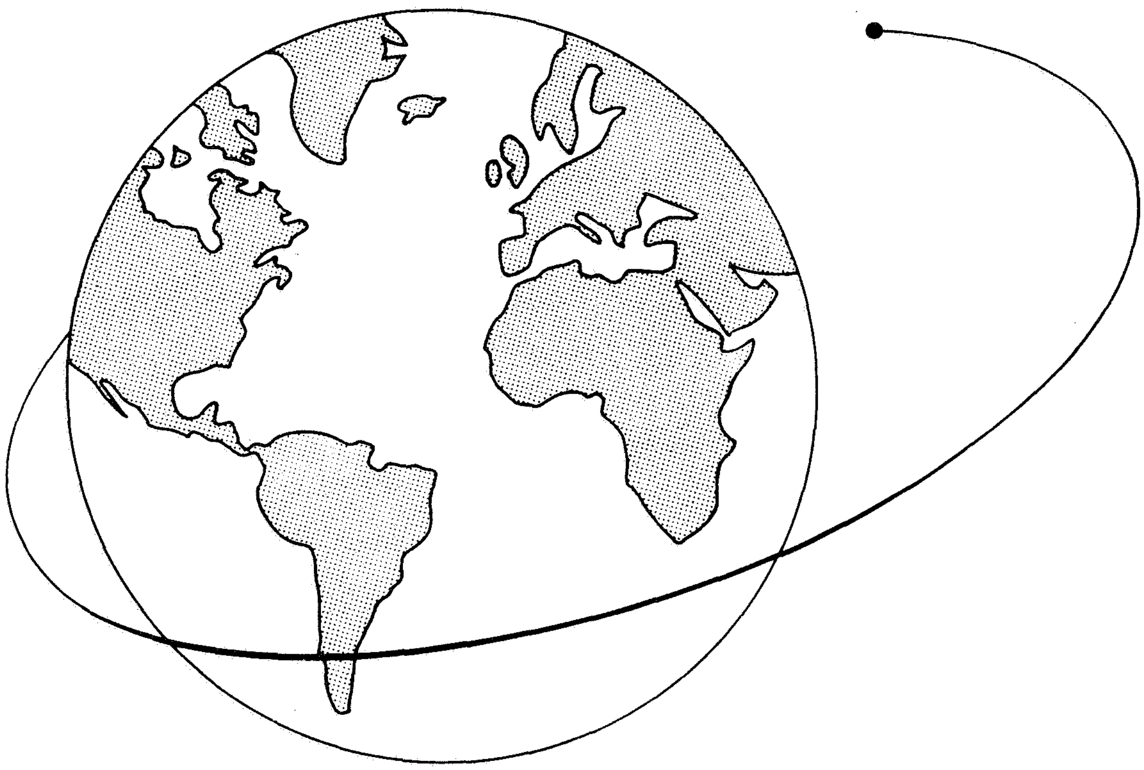
SATELLITE ALTITUDES ABOVE THE OCEAN CAN BLEND with this established practice.

Figure 2.

ATTACHMENT B

# SATELLITE ALTIMETRY AND ORBIT DETERMINATION

C. A. LUNDQUIST



Smithsonian Astrophysical Observatory
SPECIAL REPORT 248

Research in Space Science
SAO Special Report No. 248

# SATELLITE ALTIMETRY AND ORBIT DETERMINATION

Charles A. Lundquist

August 18, 1967

Smithsonian Institution
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706-3

## TABLE OF CONTENTS



## LIST OF ILLUSTRATIONS

# ABSTRACT

The technological problems associated with altimetry from a satellite are the subject of wide investigation in the United States. Therefore, it is reasonable to assume that accurate altitudes measured from satellites will eventually be available, and it is prudent to ask now what are the interfaces between altitude measurements and satellite-orbit-determination practices. From one point of view, accurate altitude data may generate accuracy requirements that must be met by orbit-determination procedures. From another point of view, the altitudes themselves may be used as tracking data in orbit determination. If the altitude of a satellite above the ocean surface is obtained, this may be viewed as a measured relationship between a point on an equipotential surface of the geopotential and a satellite position determined by the equations of motion derived from the geopotential. These various interfaces can be explored in the context of the procedures used at the Smithsonian Astrophysical Observatory (SAO) for orbit determination and geophysical research.

# RÉSUMÉ

Les problèmes technologiques associés à l'altimétrie par satellite sont l'objet de vastes investigations aux Etats-Unis. Il est par conséquent raisonnable d'admettre que des altitudes précises mesurées à partir des satellites seront finalement disponibles et il est prudent de demander maintenant quels sont les interfaces entre les mesures d'altitude et les pratiques de détermination d'orbite des satellites. D'un premier point de vue, des données précises d'altitude peuvent créer un besoin d'exactitude qui doit être satisfait par des procédés de détermination de l'orbite. D'un autre point de vue, les altitudes elles-mêmes peuvent être utilisées comme des données de poursuite dans la détermination de l'orbite. Si l'on obtient l'altitude d'un satellite au-dessus de la surface de l'océan, ceci peut être considéré comme une relation mesurée entre un point sur la surface équipotentielle du géopotentiel et une position du satellite déterminée par les équations de mouvement déduites du géopotentiel. Ces différents interfaces peuvent être explorés dans le contexte des procédés employés à l'Observatoire d'Astrophysique du Smithsonian pour la détermination des orbites et la recherche géophysique.

## Конспект

Технологические проблемы, связанные с альтиметрией, произведенной из спутника, являются объектом широкого исследования в Соединенных Штатах. Поэтому разумно предположить что точные высоты, измеренные из спутников будут со временем доступны и является благоразумным сейчас поставить вопрос о том какие существуют грани между методами измерений высоты и определения орбиты. С одной стороны, точные данные высоты могут вызывать требования точности,которая должна быть осуществлена методами определения орбиты. С другой стороны, сами высоты могут употребляться как данные наблюдений для определения орбиты. Если высота спутника над поверхностью океана получена она может рассматриваться как измеренное взаимоотношение между точкой на эквипотенциальной поверхности геопотенциала и положением спутника, определенным уравнениями движения, выведенными из геопотенциала. Эти различные грани могут быть исследованы в контексте образа действий употребляемых в Смитсониан Астрофизической Обсерватории для определения орбиты и геофизического исследования.

# SATELLITE ALTIMETRY AND ORBIT DETERMINATION

Charles A. Lundquist

## 1. INTRODUCTION AND HISTORY

Within the brief span of the space age, satellite-borne altimeters are an old idea. Interest in instrumentation to measure the altitude of a spacecraft has at least two principal motivations. One branch of activity has roots in the proposal that on-board altimeters can provide useful information to a vehicle-guidance system. A second branch stems from a desire to measure the geometrical shape of the ocean surface and its variations.

For applications near the earth, most altimeter-based guidance schemes would use the ocean surface as a reference from which to measure the space vehicle position (e. g., Godbey and Roeder, 1962; Speer and Kurtz, 1963). A similar philosophy prevails in suggestions to use an altimeter for diagnostic tracking during vehicle-development tests or critical orbital operations (e. g., Hoffman and Olthoff, 1963). For particular guidance or tracking accuracy requirements, this point of view implies that the ocean-surface geometry must be known with corresponding accuracy. Typically, space-vehicle engineers expect oceanography to provide the necessary description of sea level.

From their own point of view, various oceanographers (and geophysicists) are interested in the shape of the ocean to differing degrees of detail (Frey, Harrington, and von Arx, 1965). In the open ocean, they believe that the ocean has a static, equipotential surface to within a meter or so. Thus a

This work was supported in part by Contract NSR 09-015-054 from the National Aeronautics and Space Administration.

representation of the surface geometry to this accuracy reflects structural detail within the solid earth. For example, relative to a spheroid, sea level has about a 15-m dip in a degree of latitude across the Puerto Rico trench (von Arx, 1966). However, at decimeter accuracy, sea level varies owing to many dynamical processes — tides, cyclones, currents, etc. (Woollard, 1966). Oceanographers studying these dynamical effects assert that if instrumentation does not permit measurement of relative elevation to within 50 cm, the information is of essentially no oceanographic use (Stewart, 1965).

If satellite altimeters can approach an accuracy of 1 m, scientists concerned with deducing information about the solid earth beneath the sea become very interested; if altimetry eventually reaches decimeter accuracy, the dynamical oceanographers also become excited. In either case, they hope that practioners of celestial mechanics and satellite tracking will provide absolute satellite positions of sufficient accuracy so that the positions can be used as a reference from which to deduce sea level.

Such hopes by oceanographers on the one hand and reciprocal expectations by mechanicians on the other could carry the beginnings of a chicken-and-egg attitude toward the use of altimetry data: Which comes first, an accurate geoid or accurate orbits? The actual situation is not quite this extreme, fortunately, and several authors point out that geoid and orbit improvements can proceed together (Godbey, 1965; Frey et al., 1965; Rouse, Waite, and Walters, 1966; Lundquist, 1967). The purpose of this paper is to outline one way in which this process of mutual improvement could develop naturally. The outlined process follows the established practice in satellite geodesy.

Before a discussion of the procedures, a few background remarks about altimeter hardware are in order. The systems flown and proposed to date transmit an electromagnetic signal from the spacecraft toward the ocean surface or the solid surface from whence a reflected signal returns to the satellite. The transit time, corrected for atmosphere effects, measures the altitude of the spacecraft above the reflecting surface. The electromagnetic radiation can have radio frequency, light frequency, or some other frequency.

Radar altimeters for spacecraft are mostly an outgrowth of similar aircraft systems. However, the first satellite experience with reflections from the earth was a by-product of the swept-frequency topside sounder carried for ionosphere research on the Canadian Alouette launched on September 29, 1962 (Molozzi, 1964). In addition to returns from the ionosphere, many ionograms contained returns from the earth at frequencies above the critical frequency of the ionosphere (Chia, Doemland, and Moore, 1967; Moore, 1965). An altimeter designed for vehicle tracking flew in Saturn SA-4 in March, 1963 (Hoffman and Olthoff, 1963; Dugan, 1963). Preliminary designs for other systems are documented in more recent papers (Godbey, 1965; Frey et al., 1965; Westinghouse, 1966).

Over the ocean, the accuracy of a radar altimeter is related intimately to the reflecting character of the sea surface with its variable wave structure. Satellite measurement of the sea state — i. e., wave size — is an interesting topic, which has been widely discussed (e. g., Pierson, 1965). Perhaps it is fair to say, in summary, that altitudes over the ocean to an accuracy approaching a meter or so represent a reasonable expectation in future radar systems. But a note of caution is appropriate, because experience with range measurements between ground stations and active satellite transponders indicates that even for this case meter accuracy is difficult to obtain at radio frequencies.

Laser altimeters are a newer concept. Possible laser uses in guidance schemes are touched on briefly in several documents (e.g., Walker, 1965; Wyman, 1965). A spacecraft altimeter has been studied and experiments performed from an aircraft over the ocean (Raytheon Company, 1967).

Although no laser altimeter has flown yet in a spacecraft, a laser transmitter for a communication experiment was developed and carried on the Gemini-7 flight (Radio Corporation of America, 1965; Piland and Penrod, 1966).

Several potential laser systems, including ruby lasers for which much related experience exists, hold promise for achieving meter-accuracy altitudes when cloud cover permits. In a comparison of lasers with radars, this cloud-cover limitation is offset to some degree by the realization that ground-station satellite tracking with lasers is the only technique now routinely producing range data to meter accuracy (Plotkin, 1965; Lehr, 1966).

Altimeter applications to lunar problems are similar in many respects to earth problems, but differ in the important respect that the moon has no ocean to serve as a reference surface. For this latter reason, the lunar situation will not be considered further here other than to say that common hardware may be developed for use near the earth, moon, and other planets.

In summary, satellite altitudes above the ocean surface must be measured to an accuracy better than 10 m if these data are to be valuable. One-m accuracy is a reasonable objective to adopt for a first step (NASA, 1967), although neither a radar nor a laser system of this quality has been demonstrated.

## 2. UTILIZATION OF ALTITUDE DATA

Satellite geodesy has matured to the state where it is a recognized branch of geodetic science with its own established procedures. The conventional cycle of analysis begins at a tracking station, which measures some quantity depending on satellite position or velocity. Each observation yields an equation of condition relating orbital elements and geodetic parameters. Very many such equations are used to refine orbital elements and geodetic parameters, either simultaneously or cyclically. In these solutions the equations need not all arise from measurement of the same function of satellite position or velocity. Rather, a blend of data from various tracking systems can strengthen the solution.

My point here is that altimeter data can be blended into the same procedures with no essential change in philosophy or computer programs. The latter is particularly important since the computer programs in use by various investigators are all rather substantial, and any alternate program to use accurate altitudes will have to have comparable complexity. Another consequence is that satellite orbits and the geoid can be obtained simultaneously from altitude data, in the same way that present orbits and geopotential representations are derived together from tracking data. The accuracies are compatible also. For example, programs in advanced stages of development at the Smithsonian Astrophysical Observatory (SAO) for using laser range data are written to maintain a precision of 0.5 m.

In the paragraphs below, I outline the formulation for blending altitudes into the procedures practiced at SAO, following the development given by E. M. Gaposchkin (1966) for range observations; where possible, the notation is from the same source.

The suggested approach to altimetry adopts the assumption that sea level, averaged over wave structure, is an equipotential surface to an accuracy of approximately a meter. This is also about the accuracy that can reasonably be expected from future altimeter hardware. Thus, dynamical effects in the ocean are neglected for the present. The equipotential surface corresponding to sea level, i. e., the geoid, is represented by

$$U = \frac{GM}{r} \left\{ 1 + \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n C_{n0} \, P_{n0}(\sin \phi) \right.$$

$$\left. + \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left(\frac{a}{r}\right)^n \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right) P_{nm}(\sin \phi) \right\}$$

$$+ \frac{\omega^2 r^2}{2} \cos^2 \phi = C_0, \text{ a constant}$$

where

| | |
|---|---|
| $GM$ | = gravitational constant for the earth, |
| $C_{nm}$, $S_{nm}$ | = harmonic coefficients for the geopotential, |
| $a$ | = reference equitorial radius of earth, |
| $\omega$ | = rotational rate of earth, |
| $r$ | = geocentric radius to satellite, |
| $\phi$ | = geocentric latitude, |
| $\lambda$ | = longitude. |

In the coordinate system used for the orbit theory (essentially an inertial system), Figure 1 illustrates definitions of further notation. Note particularly that the earth rotates in this system, but the geocentric vector $\vec{s}$ to the sea surface is expressed in a space-fixed system. The corresponding vector in earth-fixed coordinates is $\vec{S}$, which is related to $\vec{s}$ by transformations $\mathcal{R}_s(\theta)\mathcal{R}(x, y)$; these specify the rotation of the earth and polar motion, respectively (Gaposchkin, 1966).

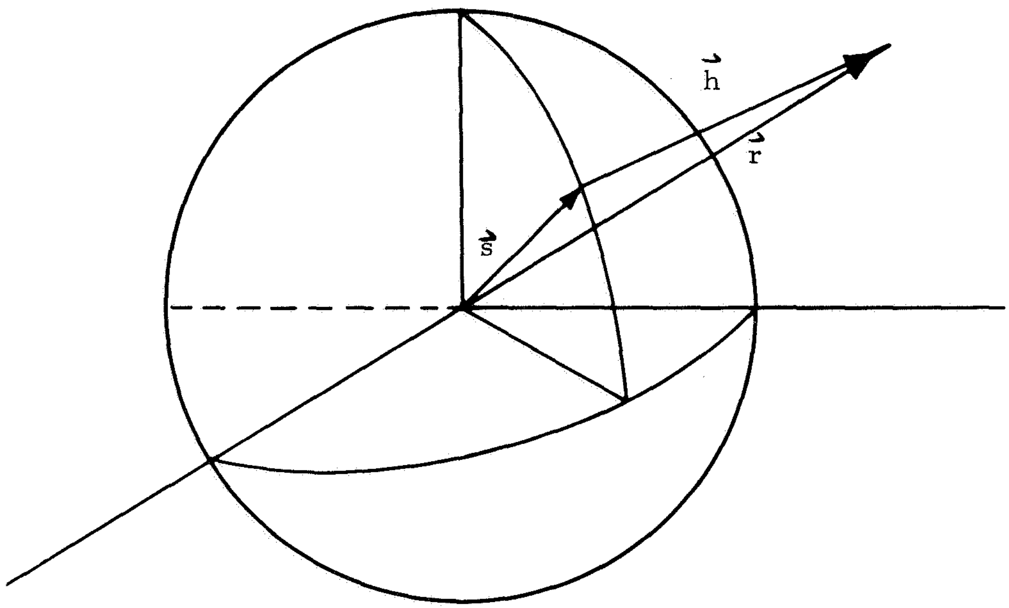


Figure 1. Relation between $\vec{r}$, $\vec{s}$, and $\vec{h}$.

where

$\vec{r}$ = satellite position,

$\vec{s}$ = position on sea surface,

$\vec{h}$ = altitude,

and

$$\vec{h}(t) = \vec{r}(t) - \vec{s}(t),$$
$$= \vec{r} - \mathcal{R}_3(\theta)\mathcal{R}(x, y)\vec{S}.$$

Further characterization of the altimeter system is necessary before proceeding. One alternative is a beam broad enough to include the point on the sea closest to the satellite. The time of the first return as sensed on the satellite gives the distance to the closest point. This system is characterized mathematically by the condition that $\vec{h}$ is normal to the geoid at $\vec{s}$. The gradient of U, rotated into the correct position at the time of observation, gives the family of normals to U, one of which contains the satellite. The resulting equations can be solved for the vector $\vec{s}$ or $\vec{S}$ as a function of $\vec{r}(t)$ and the parameters $C_i$ (= $C_{nm}$, $S_{nm}$, $C_o$) in U.

A second alternative for the altimeter system assumes a satellite with a stabilized gravity gradient, and a narrow-beam system aligned with the vehicle axis. This system is characterized by the condition that $\vec{h}$ has the direction of the gravity gradient at $\vec{r}$. Again the resulting equations can be solved for $\vec{s}$.

A third alternative would use an active attitude or pointing control on the satellite to characterize the direction of $\vec{h}$.

In all the alternate-system characterizations, the information is sufficient to determine

$$\vec{s} = \vec{s}(\vec{r}, C_i, t) \quad .$$

Since the series for U must be very long, the solution for $\vec{s}$ presumably will be performed by a computer subroutine. For the rest of the discussion here it is sufficient to know that $\vec{s}$ and its derivatives can be computed without particular trouble once an altimeter system has been selected.

For ground-station tracking (following Gaposchkin, 1966) an equation of condition is expressed as

$$A(\vec{\rho}' - \vec{\rho}) = A \frac{\partial \vec{\rho}}{\partial p_i} \Delta p_i \quad ,$$

where

$\vec{\rho}$ = calculated range vector to satellite,

A is an operator such that

$A\vec{\rho}'$ = observed positional quantity,

and

$p_i$ = parameter to be refined.

For satellite altimetry, the corresponding equation is

$$B(\vec{h}' - \vec{h}) = B \frac{\partial \vec{h}}{\partial p_i} \Delta p_i \quad ,$$

where $B\vec{h}'$ = observed altitude, $h'$. The operator B in this case may be written, in terms of computed quantities, as

$$B = \frac{\vec{h}}{h} \cdot \; ,$$

so that

$$h' - h \equiv \Delta h = \frac{\vec{h}}{h} \cdot \left( \frac{\partial \vec{h}}{\partial p_i} \right) \Delta p_i \quad .$$

If $E_i$ represents the conventional orbital elements, then the usual orbit theory used in satellite geodesy gives (Gaposchkin, 1966)

$$\vec{r} = \vec{r}(E_i, C_i, t) \quad .$$

Expanding the equations of condition gives, for ground-station tracking from position $\vec{R}$,

$$A\left[\vec{\rho}' - \vec{r}(E_i, C_i, t) + \mathcal{R}_3(\theta)\mathcal{R}(x, y)\vec{R}\right]$$

$$= A\left[\frac{\partial \vec{r}}{\partial E_i} \Delta E_i + \frac{\partial \vec{r}}{\partial C_i} \Delta C_i - \mathcal{R}_3(\theta)\mathcal{R}(x, y)\Delta\vec{R}\right] \quad ,$$

and for altimetry

$$B\left[\vec{h}' - \vec{r}(E_i, C_i, t) + \mathcal{R}_3(\theta)\mathcal{R}(x,y)\vec{s}\right]$$

$$= B\left[\left(\frac{\partial \vec{r}}{\partial E_i} - \mathcal{R}_3\mathcal{R}\frac{\partial \vec{s}}{\partial r_j}\,\frac{\partial r_j}{\partial E_i}\right)\Delta E_i\right.$$

$$\left. + \left(\frac{\partial \vec{r}}{\partial C_i} - \mathcal{R}_3\mathcal{R}\frac{\partial \vec{s}}{\partial C_i} - \mathcal{R}_3\mathcal{R}\frac{\partial \vec{s}}{\partial r_j}\,\frac{\partial r_j}{\partial C_i}\right)\Delta C_i\right] \quad .$$

The equations of condition arising from tracking and altimetry have exactly similar forms, except that station positions are not involved in the latter. All the expressions in the altimetry equation will already have been programmed for the tracking case, except the expression involving $\vec{s}$ and its derivatives. These depend upon characterization of a particular altimeter system.

From the similarity of the equations it would seem quite easy to blend altimeter observations with the other information from tracking, but some details of course need to be examined further. For example, the geoid will surely depend sensibly upon very many more harmonic coefficients than does the orbit. This is a strength of altimetry, since it will allow a more detailed representation of the geopotential. It may also create problems, if vast numbers of measured altitudes are required to obtain a reasonable solution for very many harmonic coefficients. The lack of data over continents may raise other troubles, since uniform data coverage is probably quite necessary to a uniform representation of the geopotential. In the case of continents, surface-gravity data can perhaps augment altitude data from the oceans. Kaula (1966) has already demonstrated that gravity data can be combined with satellite determinations of the geopotential. Gravity data from oceans can provide an interesting check on the results from altimetry. Fortunately, it will be quite easy to explore many of these questions by the use of the existing programs with slight modifications to simulate altimetry information.

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## BIOGRAPHICAL NOTE

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## NOTICE

This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory.

First issued to ensure the immediate dissemination of data for satellite tracking, the reports have continued to provide a rapid distribution of catalogs of satellite observations, orbital information, and preliminary results of data analyses prior to formal publication in the appropriate journals. The Reports are also used extensively for the rapid publication of preliminary or special results in other fields of astrophysics.

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ATTACHMENT C

# POSSIBLE GEOPOTENTIAL IMPROVEMENT FROM SATELLITE ALTIMETRY

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## ABSTRACT

To improve the geopotential representation, the use of satellite-to-ocean altitudes anticipates that the altimeter will be accurate to a few meters and that the open ocean approximates an equipotential surface to a few meters. Computational problems that might arise in the analyses could be largely circumvented by using a different, but equivalent, set of functions to represent the geopotential instead of using spherical harmonics. At any point on the geoid, only a very few of these alternative functions have significant values; the rest are negligible.

# POSSIBLE GEOPOTENTIAL IMPROVEMENT FROM SATELLITE ALTIMETRY*

C. A. Lundquist and G. E. O. Giacaglia**

## 1. INITIAL ASSUMPTIONS

A promising objective for an altimeter on a satellite is the collection of information about the geopotential. This is surely not the only objective, nor was it the motivation for early studies and experiments. For a brief history of spacecraft altimetry, see Lundquist (1967a). The possibility that satellite-to-ocean altimeters offer for improving the geopotential is the only subject discussed here, but this limitation by no means detracts from the importance of other objectives.

The present discussion will not be concerned with the altimeter hardware that might be selected for satellite use. An earlier study concluded that the data from any of the several possible systems could be used in essentially the same way with only minor differences in data processing details (Lundquist, 1967a). Equipment options are under study by other groups (e.g., Raytheon, 1968a, b, c). An accuracy no poorer than a few meters is assumed, however.

Use of satellite-to-ocean altitudes for improving the geopotential accepts the concept that the open ocean is an equipotential surface to an accuracy of a few meters. A recent study at New York University (Greenwood, Nathan, Neumann, Pierson, Jackson, and Pease, 1967) reviews departure of mean sea level from an equipotential surface due to various geophysical phenomena. In fine detail, at decimeter precision, the sea level depends upon many factors that are of great interest to oceanography. Thus a reliable description of sea level corresponding to an equipotential surface is a necessary step on the way toward more detailed applications of altimetry to oceanography problems.

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Among these oceanographic topics, waves deserve specific discussion. Clearly the signal returned from the ocean must be processed by the altimeter in such a way as to average over the wave structure beneath the satellite. Since ocean waves seldom exceed a few tens of meters amplitude, an averaging accuracy of 10% is all that is required for the geopotential application. This may not be a completely solved problem to date, but it seems likely to be solved. If uncertainty remains at the time an altimeter is flown on a satellite, an appropriate in-flight calibration program can certainly be established. Laser tracking from a ground station probably can provide the standard for calibration when the station is near an ocean. Hereafter the assumption is adopted that the altimeter provides altitudes properly corrected for the ocean waves, that is, for sea state, to an accuracy of a meter or so.

Other corrections must be made because an equipotential surface calculated from the expression in spherical harmonics applicable at satellite altitudes will not be correct at sea level due to the gaseous and solid mass above the geoid. Veis (1967) has discussed the atmospheric correction which has a relative magnitude of $10^{-6}$. Madden (1968) has examined the mathematical questions concerned with the fact that the sea level lies within the smallest sphere containing the solid mass of the earth. He has shown that the correction that must be applied to correct the calculated equipotential is no larger than a few meters (Madden, 1968). His work also yields the formulas to make this correction. Hereafter it will be assumed that these corrections are applied if necessary.

In summary, the discussions to follow assume that satellite-to-ocean altitudes can be measured to the accuracy with which the ocean is an equipotential surface. Quantitatively this anticipates that the altimeter will be accurate to a few meters and that the open ocean is an equipotential surface to a few meters. The former seems to be possible in the judgment of design engineers, and the latter represents the best estimate of oceanographers.

The great value of satellite-to-ocean altitudes for improving a description of the geopotential resides in the fact that sea level is much more sensitive to fine detail in the potential than is the satellite motion governed by the potential. Stated another way, knowledge of a satellite orbit to 1 m tells much less about details in the potential than does knowledge of sea level to 1-m accuracy. Nevertheless it seems likely that the altimeter data must be used concurrently to refine both the orbit and the potential.

## 2. TRACKING DATA

The conclusion was reached in previous publications that satellite-to-ocean altitudes can be treated in the same manner as other satellite-tracking data (Lundquist, 1967a, b). The arguments supporting this conclusion will not be repeated here. The referenced publications give the general expressions by which altitude observations yield equations for improving both the orbit of the satellite and the geopotential representation. The equations are identical in form to the equations employed, for example, to utilize station-to-satellite range measurements.

Altitude observations will not alone be sufficient to determine an accurate orbit for the satellite carrying the altimeter. The semimajor axis, eccentricity, and position of perigee should be well determined from the altimetry. The orbit inclination and the position of the node will depend strongly upon tracking from fixed ground stations.

Laser tracking from ground stations is likely to yield the most useful data to blend with altitudes in orbit determination and geopotential improvement. Laser ranging is particularly compatible with altimetry because both are distance measurements and because laser systems can easily measure distances to meter accuracies or better (e. g. , Plotkin, 1968; Bender, 1967; Lehr, Maestre, and Anderson, 1967). Other ground tracking data will be valuable in proportion to their ability to contribute to orbits approaching accuracies of a few meters.

The geodetic heights of tracking stations determined from conventional geodetic leveling referenced to sea level gauges can provide a valuable confirmation or calibration for the altimeter. For stations near the coast, this cross check can be accomplished through a simple calculation using the geodetic height, accurate geocentric station coordinates, and simultaneously measured satellite-to-ocean altitudes and station-to-satellite ranges. Perhaps large lakes near tracking stations could provide a similar check. The probability that the satellite pass over the lake would be greatly increased if the latitude of the lake were the same as the inclination of the satellite.

## 3. GEOPOTENTIAL DETAIL

A crucial question is the detail in the geopotential representation that may be expected from analyses of satellite-to-ocean altitudes of the accuracies anticipated. As an initial approach to this problem, present knowledge of the sea profile can be examined.

Geoid surfaces calculated from previous satellite-determined geopotentials give one indication of the expected profile. The geopotential contained in the 1966 Smithsonian Institution Standard Earth is a typical example (Gaposchkin, 1966a). This is a geopotential representation in spherical harmonics through indices 8, 8, with assorted higher degree terms. It reveals large-scale variations up to more than 100 m in the geoid position relative to a reference ellipsoid. Clearly, an altimeter accurate to a few meters should easily detect and confirm these large-scale features of sea level.

Something is also known about rather small-scale variations in sea level. For example, a shipboard measurement by von Arx (1966) has shown that in $\sim 1°$ of latitude over the Puerto Rico trench, the geoid has about a 15-m depression relative to an ellipsoid. A feature of this size would also be detected by an altimeter having an accuracy of a few meters. Although the Puerto Rico trench is unusual, it is surely not unique in the oceans. The signature in the geopotential of many such features would be obtained by a satellite that observed each square degree of the ocean.

This last remark has awesome consequences, for there are some 40,000 one-degree squares on the surface of a sphere and perhaps 25,000 of these would lie over the open ocean. To represent a geopotential that is allowed to have an arbitrary value for each one-degree square, an expansion with some 40, 000 terms and coefficients is required. From a slightly different point of view, an expansion in spherical harmonics through indices 180, 180 is appropriate to represent variations having wavelengths of one degree. Thus it seems that to represent the detail that could be detected by an altimeter, a very extensive geopotential model is implied.

Stated in a more affirmative manner, satellite-to-ocean altimetry offers the possibility of a vast improvement in the knowledge of the geopotential. Perhaps the realization of this possibility should be discussed (or even executed) in more modest steps than the jump from an 8, 8 to a 180, 180 representation.

As a first step, the possible altimeter contribution to a spherical harmonic expansion through 15, 15 can be examined. This is the detail specified as an objective in the present United States National Satellite Geodesy Program (Rosenberg, 1965). Several authors have noted that even through this number of harmonics there may be coefficients that are difficult to determine only by their effects on the orbits of typical satellites (Strange and Rainey, 1968).

An expansion through 15, 15 has a shortest wavelength of twelve degrees, and 256 harmonic coefficients. To this detail, for altitude observations, there should be no difficulty in using exactly the same procedures and computer programs conventionally employed to determine geopotential coefficients from satellite-tracking data. The pertinent equations have been discussed (Lundquist, 1967b). Over the oceans, furthermore, the altitude measurements should easily supply the observational material necessary for a sound solution, thus alleviating the mentioned deficiencies in a solution based only on ground-station tracking.

Over land areas, there may still be a deficiency in information necessary for a full 15, 15 solution since the altitude data are of no help here. However, surface gravity measurements and astrogeodetic geoids may be available in enough detail to complete the requirements for the desired solution. Procedures for using surface gravity information in combination with satellite geopotential solutions have been demonstrated (e. g., Köhnlein, 1967). The published combinations of this kind all suffer from poor distribution of gravity data. The oceans are covered particularly sparsely, but this is just where the altimetry will fill the gap.

In summary, it is quite reasonable to expect that a modest number of well-distributed satellite-to-ocean altitudes, combined with ground-station tracking, surface gravity, and astrogeodetic geoids will provide for a strong 15, 15 geopotential solution. This may well be a convenient way to complete such a solution, although the use of enough satellites in resonance with various terms in the potential might eventually complete the task (Strange and Rainey, 1968).

The next step to consider in geopotential detail is probably a solution through 36, 36, corresponding to variations down to 5-degree wavelength. This degree of detail is suggested because it is common to average surface gravity over 5-degree squares, and a combination of commensurate surface and altimetry data could be made. This enterprise would pose some problems not encountered seriously in the 15, 15 case.

The first problem arises from the number of coefficients in an expansion terminated at 36, 36; there are $37^2 = 1369$ of these coefficients. This says that the potential at any point is calculated by summing these many terms in a series. Also, the approach discussed for the 15, 15 case, if used in the 36, 36 case, would imply the inversion of a $1369 \times 1369$ matrix to solve for the coefficients. These operations are possible on modern computers, but avoiding them would be a practical advantage.

The second problem with a 36, 36 solution arises because there is little hope that the necessary data can be obtained for this detail in all parts of the globe. Over the oceans, an altimeter could provide enough observations for each $5° \times 5°$ area, but many land areas lack correspondingly dense surface measurements. Also, the orbital perturbations caused by potential terms of this degree are in general too small to be used to find the potential coefficients, although many resonance cases are an exception to this situation.

These problems could be largely circumvented by using a different, but equivalent, set of functions to represent the geopotential instead of using spherical harmonics. These alternative functions are discussed in some detail in subsequent sections. In brief, there must be the same number of independent functions—that is, 1369 functions for the equivalent of a 36,36 representation. However, at any point on the geoid, only a very few of the functions have significant values; the rest are negligible. Thus the potential at any point is the sum of only a very few terms. By adjusting the coefficients of the alternative functions it is easy to represent geopotential detail where it is known. On the other hand, short-wavelength detail can be conveniently avoided where detailed data are not available. Finally, the alternative functions are a linear combination of the spherical harmonics so that the transformation back and forth between the potential expressed in the equivalent sets of functions can be performed easily with a matrix of constant coefficients.

If a geopotential expansion through 180,180 is finally considered, so as to include such detail as those caused by the Puerto Rico trench, the problems mentioned for the 36,36 representation are further exaggerated. Here 32,761 terms in a spherical harmonic expansion contribute to the potential at each point, which is clearly an impractical way to specify the potential. Still, over the oceans an altimeter promises to measure this detail. Over land, corresponding data will very probably not be known, except perhaps in highly developed regions, such as the United States or Europe. Again, an alternative set of functions seems to be a way to plan for treatment of altimeter data.

The rest of this paper pursues this thought. No claim is made that this approach is unique. Nor has the utility of the method yet been demonstrated in practice, although typical cases could be simulated easily. The use of altimetry data seems to present no obstacles of principle, but rather to pose problems of numerical procedures. The discussions in the following sections suggest a likely solution to these problems.

## 4. FUNCTIONS OF ONE VARIABLE

The characteristics of the functions suggested as alternatives to three-dimensional spherical harmonics can first be illustrated by more simple functions of one variable (Giacaglia and Lundquist, 1968). Any function represented by a truncated Fourier series can be represented equivalently as a linear combination of these alternative functions of one variable in essentially the same way that alternative functions of three variables would replace the spherical harmonics.

To illustrate this situation, consider a real variable $0 \leq \lambda < 2\pi$, and a set of $2N + 1$ functions $(1, \cos\lambda, \sin\lambda, \cos 2\lambda, \sin 2\lambda, \ldots, \cos N\lambda, \sin N\lambda)$. These elementary trigonometric functions span a linear vector space of dimension $2N + 1$. Any function in the space can be represented in the form

$$f(\lambda) = \sum_{j=0}^{N} (c_j \cos j\lambda + s_j \sin j\lambda) \quad , \tag{4.1}$$

where $c_j$ and $s_j$ are constants. Conversely, any function representable in this form is in the space. The set $(\cos j\lambda, \sin j\lambda)$ is a basis in the space.

Another set of functions forming a basis in the same space is desired, with the property that each function $q_j(\lambda)$ have significant values for arguments near some $\lambda = \lambda_j$, $0 \leq \lambda_j < 2\pi$, and small values elsewhere in the same interval. For simplicity of language, this will be called the localized property of the function, since each is significant only in one locality in the domain of its argument.

A set of functions with this property can be generated by the requirement that

$$q_k(\lambda_j) = \delta_{kj} \quad , \quad k = 0, 1, 2, \ldots, 2N \tag{4.2}$$

for

$$\lambda_j = j \frac{2\pi}{2N+1} \quad , \quad j = 0, 1, 2, \ldots, 2N \quad .$$

Conditions (4. 2) are sufficient to determine the coefficients in equation (4. 1) for the representation of $q_i$ as a linear combination of the trigonometric functions.

Imposing conditions (4. 2) yields the relations

$$\begin{aligned} q_{Nj}(\lambda) &= \frac{1}{2N+1} \left[ 1 + 2 \sum_{\ell=1}^{k} \cos \ell \, (\lambda - \lambda_{Nj}) \right] \quad , \\ \lambda_{Nj} &= j \frac{2\pi}{2N+1} \quad , \\ j &= 0, 1, 2, \ldots, 2N \quad , \\ 0 &\leq \lambda < 2\pi \quad . \end{aligned} \tag{4.3}$$

The index N is added to keep track of the dimension (2N + 1) of the space in which the $q_{Nj}$ have been generated. The inverse of (4. 3) is

$$\left.\begin{aligned} \cos m\lambda &= \sum_{j=0}^{2N} \cos m\lambda_{Nj} \, q_{Nj} \quad , \\ \sin m\lambda &= \sum_{j=0}^{2N} \sin m\lambda_{Nj} \, q_{Nj} \quad . \end{aligned}\right\} \quad m \leq N \tag{4.4}$$

With equation (4.4), any function of the form (4.1) can immediately be rewritten as a completely equivalent linear combination of the $q_{Nj}$; thus the first essential property of the $q_{Nj}$ has been demonstrated. In addition, the functions have many other useful properties.

Evaluation of equation (4.4) for m = 0 yields the fact that

$$\sum_{j=0}^{2N} q_{Nj}(\lambda) = 1 \tag{4.5}$$

The $q_{Nj}(\lambda)$ are orthogonal functions, that is,

$$\int_0^{2\pi} q_{Ni}(\lambda)\, q_{Nj}(\lambda)\, d\lambda = \frac{2\pi}{2N+1}\, \delta_{ij} \quad . \tag{4.6}$$

Also,

$$\sum_{\ell=0}^{2N} q_{Ni}(\lambda_{N\ell})\, q_{Nj}(\lambda_{N\ell}) = \delta_{ij} \quad . \tag{4.7}$$

From the form of (4.3) it is clear that all of the $q_{Nj}$ have the same shape. They differ only by translations in multiples of $2\pi/(2N+1)$ along the $\lambda$ axis:

$$q_{Nj}(\lambda) = q_{N,0}(\lambda - \lambda_{Nj}) \quad . \tag{4.8}$$

Finally, the localized property of the $q_{Nj}$ must be demonstrated. For each one of the 2N + 1 values of $\lambda$ equally spaced between 0 and $2\pi$, condition (4.2) assures that all but one of the $q_{Nj}$ are zero. Hence a particular $q_{Np}$ has a maximum value 1 at point $\lambda_{Np}$ and zero at all the other 2N points. Another way to state this is that at each of the 2N + 1 points, only the one function with the same indices contributes to the value of any linear combination of the $q_{Np}$.

Further, the function $q_{Ni}$ remains relatively small near $\lambda_{Nj}$, $i \neq j$. This is illustrated for N = 15 in Figure 1. Note that the secondary maxima of $q_{15,j}$ are substantially lower than the primary maximum. This is another manifestation of the localized property.

Closely related to the localized property is the property that if $f(\lambda)$ is in the space spanned by the $q_{Nj}(\lambda)$, then

$$f(\lambda) = \sum_{j=0}^{2N} f(\lambda_{Nj})\, q_{Nj}(\lambda) \quad . \tag{4.9}$$

Equation (4.4) is a special case of this general formula. If $f(\lambda)$ is not in the space spanned by the $q_{Nj}(\lambda)$, then the right side of (4.9) gives the projection of $f(\lambda)$ on the space.

The relations given in this section and the procedures suggested are, of course, equivalent to numerical Fourier analysis. Thus nothing essentially new has been accomplished. However, this formulation is useful as a guide to the subsequent treatment of the two- and three-dimensional cases.

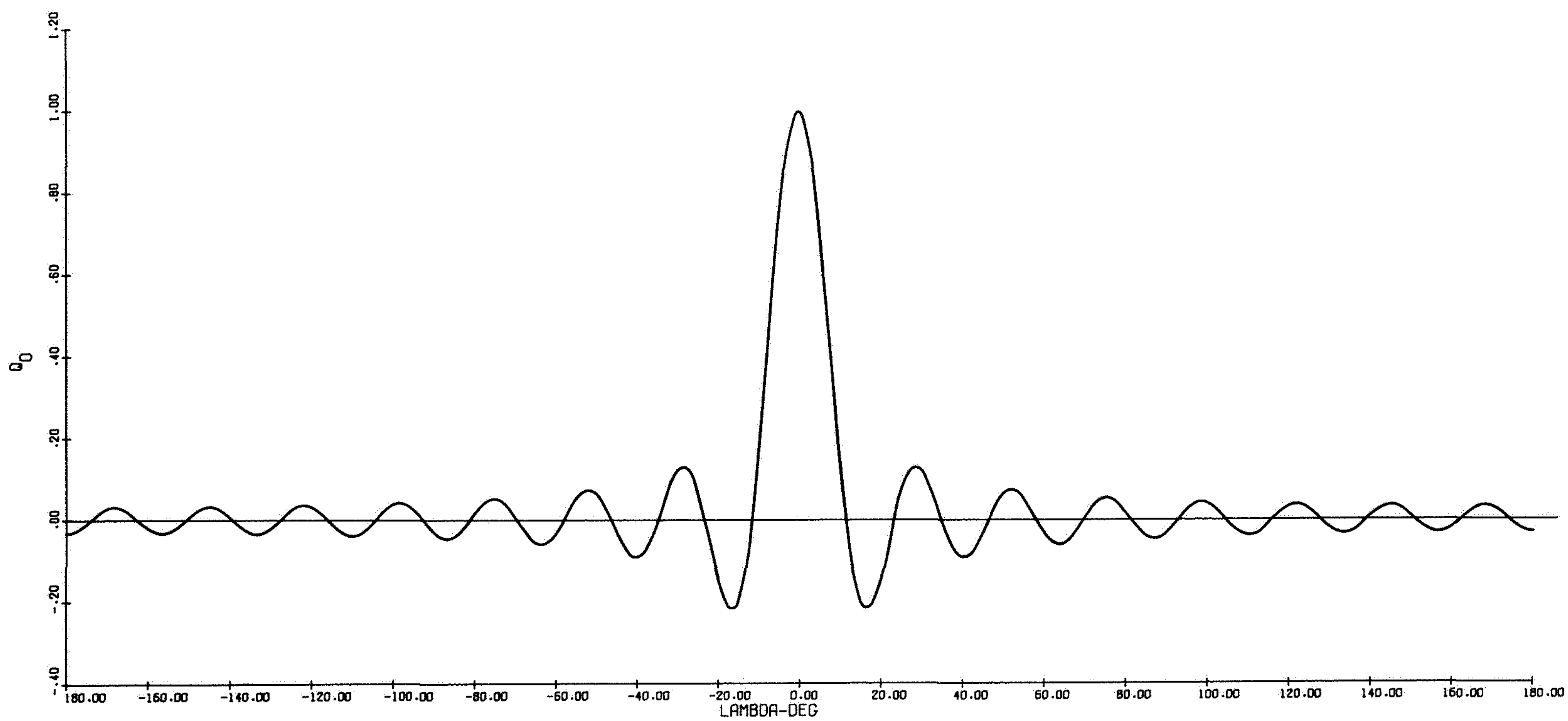


Figure 1. Graph of the function $q_{15,0}$.

## 5. FUNCTIONS ON A SPHERE

The usual surface harmonics defined on a sphere have the expressions

$$X_{nm}(\theta, \lambda) = P_{nm}(\theta) \cos m\lambda \quad , \quad (n = 0, 1, \ldots, N;\ m = 0, 1, \ldots, n) \quad ,$$

$$Y_{nm}(\theta, \lambda) = P_{nm}(\theta) \sin m\lambda \quad , \quad (n = 1, 2, \ldots, N;\ m = 1, 2, \ldots, n) \quad , \tag{5.1}$$

where $P_{nm}(\theta)$ are associated Legendre functions. If the set of functions is truncated at $n = N$, there are $M = (N + 1)^2$ independent functions in the set.

By analogy to the one-dimensional example, it is desired to define another set of M independent functions $Z_j(\theta, \lambda)$ spanning the same space as the surface harmonics and having localized properties generated by the condition that

$$Z_j(\theta_i, \lambda_i) = \delta_{ij} \quad , \quad i, j = 1, 2, \ldots, M \tag{5.2}$$

for M points, $\theta_i$, $\lambda_i$, on the sphere. From these requirements follow immediately the relations analogous to (4.4):

$$X_{nm}(\theta, \lambda) = \sum_{j=1}^{M} X_{nm}(\theta_j, \lambda_j)\ Z_j(\theta, \lambda) \quad ,$$

$$Y_{nm}(\theta, \lambda) = \sum_{j=1}^{M} Y_{nm}(\theta_j, \lambda_j)\ Z_j(\theta, \lambda) \quad . \tag{5.3}$$

Since the $Z_j$ span the same space as the $X_{nm}$, $Y_{nm}$, the former must also be expressible as linear combinations of the latter. Thus

$$Z_j(\theta, \lambda) = \sum_{n=0}^{N} \sum_{m=0}^{n} \left[A_j^{nm} X_{nm}(\theta, \lambda) + B_j^{nm} Y_{nm}(\theta, \lambda)\right] \quad . \tag{5.4}$$

The constants $A_j^{nm}$ and $B_j^{nm}$ can be determined either from conditions (5.2) and equation (5.4) or from inverting equation (5.3). The two operations are equivalent.

One important property of the $Z_j$ follows directly from writing (5.3) for $n = m = 0$. Since $X_{00}(\theta, \lambda) = 1$, equation (5.3) gives

$$\sum_{j=1}^{M} Z_j(\theta, \lambda) = 1 \quad . \tag{5.5}$$

Equations (5.3) are implicit definitions of the functions $Z_j(\theta, \lambda)$ once a set of points $\theta_j$, $\lambda_j$ have been selected on the sphere. The equations (5.3) are conceptually simple, but the selection of points seems to be a more profound topic.

First the points must be distributed in such a way that the matrix corresponding to the coefficients in equation (5.3) is not singular. That is,

$$\begin{vmatrix} X_{00}(\theta_1, \lambda_1) & X_{00}(\theta_2, \lambda_2) & \cdots & X_{00}(\theta_M, \lambda_M) \\ X_{10}(\theta_1, \lambda_1) & X_{10}(\theta_2, \lambda_2) & \cdots & X_{10}(\theta_M, \lambda_M) \\ & \cdots & & \\ Y_{11}(\theta_1, \lambda_1) & Y_{11}(\theta_2, \lambda_2) & \cdots & Y_{11}(\theta_M, \lambda_M) \\ & \cdots & & \\ Y_{NN}(\theta_1, \lambda_1) & Y_{NN}(\theta_2, \lambda_2) & \cdots & Y_{NN}(\theta_M, \lambda_M) \end{vmatrix} \neq 0 \quad . \tag{5.6}$$

Clearly, also, it is desirable to distribute the $M = (N + 1)^2$ points uniformly over the sphere in some sense. Of course, there are but five regular polyhedra, and only the tetrahedron has a number of vertices equal to the square of an integer. Hence the distribution must depart somehow from ideal regularity.

Numerical experiments or general arguments show further that a distribution that is symmetrical with respect to the equatorial plane and has rotational symmetry about the polar axis yields a zero value for the determinant (5.6). For example, if $N = 3$, a distribution with one point at each pole and seven uniformly spaced points at 30°N and 30°S latitude is not an acceptable array of points, even though it is an attractively uniform distribution of the correct number of points.

If one selects an arbitrary distribution of points satisfying (5.6), then (5.3) and its inverse by mathematical brute force are virtually the end of the story, unless particular properties of the distribution allow further conclusions to be reached. The cases in which (5.3) must be inverted numerically to express the $Z_i$ as combinations of the $X_{nm}$, $Y_{nm}$ can already be quite useful, since the inversion need be done only once to obtain a usable expression for the $Z_i$.

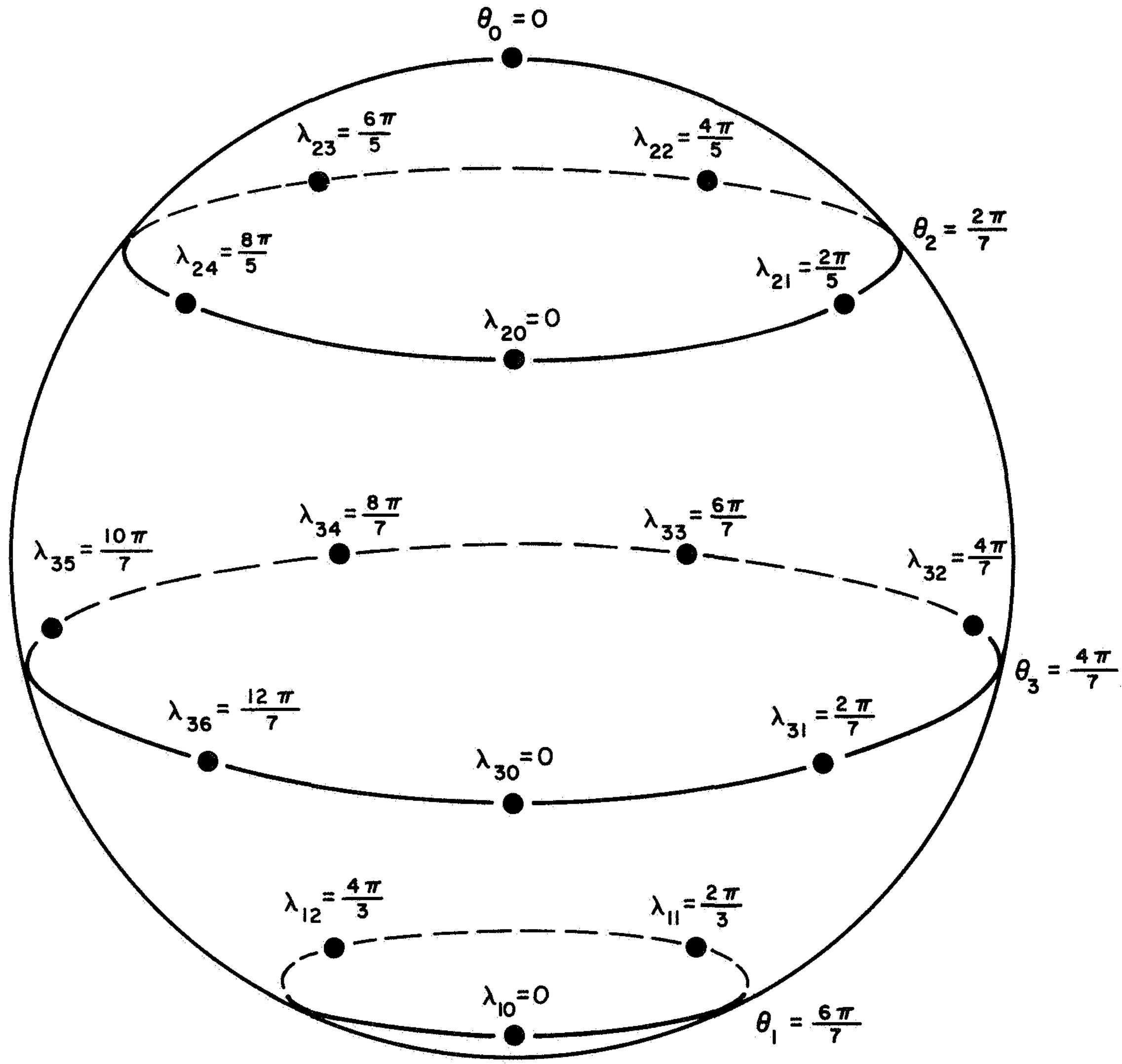


Figure 2. Distribution of points on the sphere for N = 3.

## 6. THE FUNCTIONS $W_{kj}(\theta, \lambda)$

There is at least one distribution of points on a sphere for which much can be proved about the properties of the $Z_j(\theta, \lambda)$. This distribution recognizes that there are some natural requirements on the point distribution suggested by the wavelengths inherent in the surface harmonics. Thus there are $2N + 1$ independent functions of longitude involved in the surface harmonics, and it is natural to expect that there be $2N + 1$ distinct longitudes in the point distribution. Similarly one might expect $(2N + 1)/2$ divisions of latitude between the poles.

First, define $N + 1$ latitude values by the equations

$$\theta_{2i} = \frac{i2\pi}{2N+1} \quad , \quad \left(i = 0, 1, \dots, \frac{N}{2}\right)$$

$$\theta_{2i+1} = \frac{(N-i)2\pi}{2N+1} \quad . \quad \left(i = 0, 1, \dots, \frac{N-1}{2}\right) \tag{6.1}$$

Figure 2 illustrates the spacing of these points and the indexing convention. Next, at each of these latitudes, $\theta_k$, define $2k + 1$ equally spaced points in longitude:

$$\lambda_{kj} = \frac{j2\pi}{2k+1} \qquad (j = 0, 1, \dots, 2k) \tag{6.2}$$

These are also illustrated in Figure 2.

For any positive integer N, equations (6.1) and (6.2) define $(N + 1)^2$ points. There are $N + 1$ latitudes, starting at the North Pole. The spacing in latitude between circles of points is $2\pi/(2N + 1)$, except that the most southerly latitude is only half this distance from the South Pole. This most southerly latitude circle has three points. Below the single point at the North Pole, the next circle has five points. The second most southerly circle has seven points, etc. The latitude circle nearest the Equator has the greatest number of points, namely, $2N+1$. Also, any meridian circle has $2N+1$ uniformly spaced intersections with the latitude circles. In this sense, the points are spaced equally in latitude and longitude. For notational brevity, this distribution of points will sometimes be referred to as the 1-3-5 ... distribution.

An important property of this distribution is that condition (5.6) is satisfied, and the $Z_j$ are well defined. For this particular point distribution, the functions $Z_j(\theta, \lambda)$ will be denoted by $W_{kj}(\theta, \lambda)$, where

$$W_{kj}(\theta_\ell, \lambda_{\ell i}) = \delta_{kj, \ell i} \quad . \tag{6.2}$$

Also,

$$W_{kj}(\theta, \lambda) = \sum_{n=0}^{N} \sum_{m=0}^{n} \left[A_{kj}^{nm} X_{nm}(\theta, \lambda) + B_{kj}^{nm} Y_{nm}(\theta, \lambda)\right] \quad , \tag{6.3}$$

and

$$X_{nm}(\theta, \lambda) = \sum_{k=0}^{N} \sum_{j=0}^{2k} X_{nm}(\theta_k, \lambda_{kj}) W_{kj}(\theta, \lambda) \quad ,$$

$$Y_{nm}(\theta, \lambda) = \sum_{k=0}^{N} \sum_{j=0}^{2k} Y_{nm}(\theta_k, \lambda_{kj}) W_{kj}(\theta, \lambda) \quad . \tag{6.4}$$

From (5.5) or (6.4) it follows that

$$\sum_{k=0}^{N} \sum_{j=0}^{2k} W_{kj}(\theta, \lambda) = 1 \quad . \tag{6.5}$$

A number of further properties of the $W_{kj}$ can also be proved. Consider initially $k \neq 0$ and let $\theta_k$ not be a zero of $P_{nm}(\theta)$, whatever are the values of n and m. This is true for the 1-3-5 ··· distribution. It follows that

$$W_{kj}(\theta_k, \lambda) = \sum_{m=0}^{N} \left\{ \left[ \sum_{n=m}^{N} A_{kj}^{nm} P_{nm}(\cos \theta_k) \right] \cos m\lambda + \right.$$

$$\left. + \left[ \sum_{n=m}^{N} B_{kj}^{nm} P_{nm}(\cos \theta_k) \right] \sin m\lambda \right\}$$

$$= \sum_{m=0}^{N} \left( a_{kj}^{m} \cos m\lambda + b_{kj}^{m} \sin m\lambda \right) \tag{6.6}$$

But equation (6.2) requires for a fixed k that

$$W_{kj}(\theta_k, \lambda_{k\ell}) = \delta_{j\ell} \quad . \tag{6.7}$$

Equations (6.6) and (6.7) define uniquely a set of $2k + 1$ functions of $\lambda$, namely,

$$W_{kj}(\theta_k, \lambda) = q_{kj}(\lambda) = \frac{1}{2k+1} \left[ 1 + 2 \sum_{m=1}^{k} \cos m(\lambda - \lambda_{kj}) \right] \quad . \tag{6.8}$$

$$j = 0, 1, \ldots, 2k$$

The $q_{kj}(\lambda)$ are identically the functions discussed in Section 4. Thus for each latitude circle defined by $\theta_k$, the $W_{kj}$ evaluated for $\theta = \theta_k$ reduce to the $q_{kj}(\lambda)$. This is one sense in which the $W_{kj}$ are natural extensions of the $q_{kj}$.

To illustrate another related property of the $W_{kj}$, it is convenient first to define a further set of functions closely related to the $q_{kj}$:

$$C_0(\theta) = \frac{1}{2N+1}\left(1 + 2\sum_{j=1}^{N} \cos j\theta\right) \quad ,$$

$$C_\ell(\theta) = \frac{2}{2N+1}\left(1 + 2\sum_{j=1}^{N} \cos j\theta_\ell \, \cos j\theta\right) \quad , \tag{6.9}$$

$$\theta_{2i} = \frac{i2\pi}{2N+1} \quad , \quad (i = 0, 1, \ldots, \frac{N}{2})$$

$$\theta_{2i+1} = \frac{(N-i)2\pi}{2N+1} \quad . \qquad (i = 0, 1, \ldots, \frac{N-1}{2}) \tag{6.10}$$

These functions have the properties that

$$C_\ell(\theta_j) = \delta_{\ell j} \quad , \quad j, \ell = 0, 1, \ldots, N \quad . \tag{6.11}$$

$$\sum_{\ell=0}^{N} C_\ell(\delta) = 1 \tag{6.12}$$

Conversely, if $C_\ell(\theta)$ is a linear combination of functions $\cos j\theta$, point distribution (6.10) and conditions (6.11) and (6.12) imply equations (6.9). The functions $C_\ell$ thus are functions of latitude having localized properties similar to those of the $q_{kj}$.

Now for a fixed k in equation (6.3), form the sum of the $W_{kj}(\theta, \lambda)$ for the values of j, $(j = 0, 1, \ldots, 2k)$. This can be shown to yield the result

$$\sum_{j=0}^{2k} W_{kj}(\theta, \lambda) = C_k(\theta) \quad , \tag{6.13}$$

and, in particular,

$$W_{00}(\theta, \lambda) = C_0(\theta) \quad . \tag{6.14}$$

Equations (6.13) and (6.14) are another manifestation of the localized properties of the $W_{kj}$.

## 7. EXAMPLES OF $W_{kj}(\theta, \lambda)$

The $W_{kj}(\theta, \lambda)$ for any N are implicitly defined by equations (6.4). For small values of N it is possible to write explicit expressions for the $W_{kj}$ in simple form.

For N = 1, the functions are:

$$W_{00} = \frac{1}{3}(1 + 2\cos\theta) \quad ,$$

$$W_{1j} = \frac{2}{9}(1 + 2\cos\theta_1 \cos\theta) + \frac{2\sin\theta}{3\sin\theta_1}\cos(\lambda - \lambda_{1j}) \quad , \tag{7.1}$$

$$\text{for } j = 0, 1, 2 \quad ,$$

where

$$\theta_0 = 0 \quad ,$$

$$\theta_1 = \frac{2\pi}{3} \quad ; \quad \lambda_{10} = 0, \; \lambda_{11} = \frac{2\pi}{3} \quad , \quad \lambda_{12} = \frac{4\pi}{3} \quad .$$

For N = 2, the functions are:

$$W_{00} = \frac{1}{5}(1 + 2\cos\theta + 2\cos 2\theta) \quad ,$$

$$W_{1j} = \frac{2}{15}(1 + 2\cos\theta_1 \cos\theta + 2\cos 2\theta_1 \cos 2\theta)$$
$$+ \frac{2\sin\theta(\cos\theta_2 - \cos\theta)}{3\sin\theta_1(\cos\theta_2 - \cos\theta_1)}\cos(\lambda - \lambda_{1j}) \quad , \quad (j = 0, 1, 2)$$

$$W_{2j} = \frac{2}{25}(1 + 2\cos\theta_2 \cos\theta + 2\cos 2\theta_2 \cos 2\theta)$$

$$+ \frac{2\sin\theta_2 \sin\theta(\cos\theta - \cos\theta_1)}{5\sin^2\theta_2(\cos\theta_2 - \cos\theta_1)} \cos(\lambda - \lambda_{2j})$$

$$+ \frac{2\sin\theta_1 \sin\theta(\cos\theta - \cos\theta_2)}{5\sin^2\theta_2(\cos\theta_2 - \cos\theta_1)} \cos(\lambda + 2\lambda_{2j})$$

$$+ \frac{2\sin^2\theta}{5\sin^2\theta_2} \cos 2(\lambda - \lambda_{2j}) \quad , \qquad (j = 0, 1, 2, 3, 4)$$

where

$$\theta_0 = 0 \quad ,$$

$$\theta_1 = \frac{4\pi}{5} \quad ; \quad \lambda_{10} = 0, \ \lambda_{11} = \frac{2\pi}{3}, \ \lambda_{12} = \frac{4\pi}{3} \quad ,$$

$$\theta_2 = \frac{2\pi}{5} \quad ; \quad \lambda_{20} = 0, \ \lambda_{21} = \frac{2\pi}{5}, \ \lambda_{22} = \frac{4\pi}{5}, \ \lambda_{23} = \frac{6\pi}{5}, \ \lambda_{24} = \frac{8\pi}{5} \quad .$$

Numerical examples for larger values of N are given by Hebb and Mair (1968).

## 8. FUNCTIONS IN THREE DIMENSIONS

In three dimensions the functions in which the potential is usually expanded can be written as

$$\mathcal{X}_{nm}(r,\theta,\lambda) = R_n(r)\,X_{nm}(\theta,\lambda) \quad ,$$

$$\mathcal{Y}_{nm}(r,\theta,\lambda) = R_n(r)\,Y_{nm}(\theta,\lambda) \quad , \tag{8.1}$$

where the $X_{nm}$ and $Y_{nm}$ are the surface harmonics in equation (5.1) and

$$R_n(r) = \left(\frac{a}{r}\right)^{n+1} \quad . \tag{8.2}$$

There are $M = (N+1)^2$ functions (8.1) in a space that corresponds to a maximum value $n = N$.

As with the one- and two-dimensional cases, it is again desired to define an alternative set of functions $\mathcal{Z}_j(r,\theta,\lambda)$ that have localized properties generated by the condition that

$$\mathcal{Z}_i(r_j,\theta_j,\lambda_j) = \delta_{ij} \quad , \tag{8.3}$$

where $(r_j,\theta_j,\lambda_j)$ is a point in a selected set of $(N+1)^2$ points. It follows immediately that

$$\mathcal{X}_{nm}(r,\theta,\lambda) = \sum_{j=1}^{M} \mathcal{X}_{nm}(r_j,\theta_j,\lambda_j)\,\mathcal{Z}_j(r,\theta,\lambda) \quad ,$$

$$\mathcal{Y}_{nm}(r,\theta,\lambda) = \sum_{j=1}^{M} \mathcal{Y}_{nm}(r_j,\theta_j,\lambda_j)\,\mathcal{Z}_j(r,\theta,\lambda) \quad . \tag{8.4}$$

Let the inverse of this be written as

$$\mathcal{Z}_j(r,\theta,\lambda) = \sum_{n=0}^{N} \sum_{m=0}^{n} \left[ \mathcal{A}_j^{nm} \mathcal{X}_{nm}(r,\theta,\lambda) + \mathcal{B}_j^{nm} \mathcal{Y}_{nm}(r,\theta,\lambda) \right] . \tag{8.5}$$

Analogously to equations (5. 3), equations (8. 4) are an implicit definition for the $\mathcal{Z}_j(r, \theta, \lambda)$. Again, the crucial question is the distribution of points $(r_j, \theta_j, \lambda_j)$. For the $\mathcal{Z}_j(r, \theta, \lambda)$ to be well defined, it is necessary and sufficient that

$$\begin{vmatrix} \mathcal{X}_{00}(r_1,\theta_1,\lambda_1) & \mathcal{X}_{00}(r_2,\theta_2,\lambda_2) & \cdots & \mathcal{X}_{00}(r_M,\theta_M,\lambda_M) \\ \mathcal{X}_{10}(r_1,\theta_1,\lambda_1) & \mathcal{X}_{10}(r_2,\theta_2,\lambda_2) & \cdots & \mathcal{X}_{10}(r_M,\theta_M,\lambda_M) \\ & & \cdots & \\ \mathcal{Y}_{11}(r_1,\theta_1,\lambda_1) & \mathcal{Y}_{11}(r_1,\theta_1,\lambda_1) & \cdots & \mathcal{Y}_{11}(r_M,\theta_M,\lambda_M) \\ & & \cdots & \\ \mathcal{Y}_{NN}(r_1,\theta_1,\lambda_1) & \mathcal{Y}_{NN}(r_2,\theta_2,\lambda_2) & \cdots & \mathcal{Y}_{NN}(r_M,\theta_M,\lambda_M) \end{vmatrix} \neq 0 . \tag{8.6}$$

If all the points $(r_j, \theta_j, \lambda_j)$ are distributed on a sphere of some radius $r_j = r_0$, then condition (8. 6) reduces immediately to (5. 6), and all of the discussions of Sections 5, 6, and 7 are pertinent to the three-dimensional case. In this situation, equations (8.4) become

$$\mathcal{X}_{nm}(r,\theta,\lambda) = \left(\frac{a}{r_0}\right)^{n+1} \sum_{j=1}^{M} X_{nm}(\theta_j, \lambda_j)\, \mathcal{Z}_j(r,\theta,\lambda) ,$$

$$\mathcal{Y}_{nm}(r,\theta,\lambda) = \left(\frac{a}{r_0}\right)^{n+1} \sum_{j=1}^{M} Y_{nm}(\theta_j, \lambda_j)\, \mathcal{Z}_j(r,\theta,\lambda) , \tag{8.7}$$

or

$$\left(\frac{r_0}{r}\right)^{n+1} X_{nm}(\theta, \lambda) = \sum_{j=1}^{M} X_{nm}(\theta_j, \lambda_j) \mathcal{J}_j(r, \theta, \lambda) \quad .$$

$$\left(\frac{r_0}{r}\right)^{n+1} Y_{nm}(\theta, \lambda) = \sum_{j=1}^{M} Y_{nm}(\theta_j, \lambda_j) \mathcal{J}_j(r, \theta, \lambda) \quad . \tag{8.8}$$

Equations (8.8) have the same form as equation (5.3), so their inverse can be expressed simply in terms

$$\mathcal{J}_j(r, \theta, \lambda) = \sum_{n=0}^{M} \sum_{m=0}^{n} \left[a_j^{nm} \left(\frac{r_0}{r}\right)^{n+1} X_{nm}(\theta, \lambda) + b_j^{nm} \left(\frac{r_0}{r}\right)^{n+1} Y_{nm}(\theta, \lambda)\right] \tag{8.9}$$

where the $a_j^{nm}$ and $b_j^{nm}$ are identical to those of Sections 5, 6, and 7 if the same points on the sphere are selected. This form is convenient for investigating the properties of the $\mathcal{J}_j(r, \theta, \lambda)$.

Let $r = r_0 + h$, so that

$$\left(\frac{r_0}{r}\right)^{n+1} = \left(\frac{r_0}{r}\right)\left(\frac{r_0}{r_0 + h}\right)^{n} = \left(\frac{r_0}{r}\right)\left(1 + \frac{h}{r_0}\right)^{-n} \quad . \tag{8.10}$$

Expanding this in a series gives

$$\left(\frac{r_0}{r}\right)^{n+1} = \left(\frac{r_0}{r}\right)\left[1 + (-n)\left(\frac{h}{r_0}\right) + \frac{(-n)(-n-1)}{2}\left(\frac{h}{r_0}\right)^2 + \dots\right] \quad . \tag{8.11}$$

Substitution of (8. 11) into (8. 9) then gives

$$\mathcal{J}_j(r, \theta, \lambda) = \sum_{n=0}^{M} \sum_{m=0}^{n} \left\{ \left(\frac{r_0}{r}\right) \left[a_j^{nm} X_{nm} + b_j^{nm} Y_{nm}\right] \right.$$

$$+ \left(\frac{r_0}{r}\right) \left(\frac{h}{r_0}\right) \left[a_j^{nm}(-n) X_{nm} + b_j^{nm}(-n) Y_{nm}\right]$$

$$+ \left(\frac{r_0}{h}\right) \left(\frac{h}{r_0}\right)^2 \left[a_j^{nm} \frac{(-n)(-n - 1)}{2} X_{nm} + b_j^{nm} \frac{(-n)(-n - 1)}{2} Y_{nm}\right]$$

$$\left. + \dots \right\} \quad . \tag{8. 12}$$

But the first term on the right of (8. 12) can be expressed simply in terms of the $Z_j$ as defined in (5. 4). Define $\mathcal{J}_j^0(r, \theta, \lambda)$ by

$$\mathcal{J}_j^0(r, \theta, \lambda) = \left(\frac{r_0}{r}\right) Z_j(\theta, \lambda) \quad . \tag{8. 13}$$

Then

$$\mathcal{J}_j(r, \theta, \lambda) = \mathcal{J}_j^0(r, \theta, \lambda)$$

$$+ \left(\frac{h}{r_0 + h}\right) \sum_{n=0}^{M} \sum_{m=0}^{n} \left[a_j^{nm}(-n) X_{nm} + b_j^{nm}(-n) Y_{nm}\right]$$

$$+ \dots \quad .$$

Since the $\mathcal{J}_j$ will be evaluated on the geoid, a value of $r_0$ can be selected such that

$$\frac{h}{r_0 + h} \sim -\frac{1}{2}\,\frac{1}{298} \text{ at the poles, and}$$

$$\frac{h}{r_0 + h} \sim +\frac{1}{2}\,\frac{1}{298} \text{ at the Equator.}$$

From the expanded form of $\mathcal{J}_j(r, \theta, \lambda)$, it can be seen that, for small values of $(h/r)$, the function retains the localized properties imposed by the factor $Z_j$ in the dominant term $\mathcal{J}_j^0$. If the 1-3-5... distribution is selected, then the $Z_j$ becomes $W_{jk}$, and still more can be said about the behavior of the $\mathcal{J}_j$.

However, there may be advantages in selecting a distribution of points not lying on a sphere. For example, the ellipsoid best approximating the geoid might be chosen, and points distributed on it according to the 1-3-5... distribution. In this case, the terms on the right of equations (8.4) have different factors $(a/r_j)^{n+1}$ because the $r_j$ are different. Still, from general considerations, it seems that the first terms accounting for displacements of the geoid from the ellipsoid will be proportional to $(h/r_R)$, where $r_R$ is on the reference ellipsoid and h is the distance from the ellipsoid. The maximum value of h for sea level will be less than 100 m, so $(h/r_R) \sim 10^{-5}$.

## 9. COEFFICIENTS OF $\mathcal{J}_j$ IN THE GEOPOTENTIAL

In spherical harmonics, the geopotential can be represented by

$$V(r, \theta, \lambda) = GM \sum_{n=0}^{N} \sum_{m=0}^{n} \left[ C_{nm} \mathcal{X}_{nm}(r, \theta, \lambda) + S_{nm} \mathcal{Y}_{nm}(r, \theta, \lambda) \right] \quad . \qquad (9.1)$$

where GM is the gravitational constant for the earth. For objects constrained to rotate with the earth, such as the oceans, the centrifugal potential must be added to this, namely,

$$+ \frac{\omega^2 r^2}{2} \sin^2 \theta \quad , \qquad (9.2)$$

where $\omega$ is the rotational rate of the earth.

Substituting expressions for $\mathcal{X}_{nm}$ and $\mathcal{Y}_{nm}$ into (9.1) gives

$$V(r,\theta,\lambda) = GM \sum_{j=1}^{M} \left\{ \sum_{n=0}^{N} \sum_{m=0}^{n} C_{nm} \mathcal{X}_{nm}(r_j, \theta_j, \lambda_j) + S_{nm} \mathcal{Y}_{nm}(r_j, \theta_j, \lambda_j) \right\} \mathcal{J}_j(r, \theta, \lambda) \quad . \qquad (9.3)$$

If the coefficient $L_j$ is defined by

$$L_j = \sum_{n=0}^{N} \sum_{m=0}^{n} \left[ C_{nm} \mathcal{X}_{nm}(r_j, \theta_j, \lambda_j) + S_{nm} \mathcal{Y}_{nm}(r_j, \theta_j, \lambda_j) \right] \quad , \qquad (9.4)$$

then

$$V(r, \theta, \lambda) = GM \sum_{j=1}^{M} L_j \mathcal{J}_j(r, \theta, \lambda) \quad . \qquad (9.5)$$

This is a representation of the geopotential that is completely equivalent to equation (9.1). However, each term in the sum dominates in one local region of the geoid, namely, the region around one of the selected grid of points. The functions $\mathcal{J}_j$ were defined so as to produce this situation. Determining the coefficients $L_j$ is equivalent to determining the coefficients $C_{nm}$ and $S_{nm}$, since the transformation and its inverse from $\mathcal{J}_j$ to $\mathcal{X}_{nm}$ and $\mathcal{Y}_{nm}$ also define the transformations between the coefficients.

In the space around the earth, the equation of an equipotential surface is

$$GM \sum_{j=1}^{M} L_j \mathcal{J}_j(r, \theta, \lambda) - V = 0 \quad , \tag{9.6}$$

where V is a constant. The geoid, that is, mean sea level, is given by

$$GM \sum_{j=1}^{M} L_j \mathcal{J}_j(r, \theta, \lambda) + \frac{\omega^2 r^2}{2} \sin^2 \theta - V_0 = 0 \quad . \tag{9.7}$$

There are several ways that a determination of the geopotential might proceed utilizing the localized properties of the $\mathcal{J}_j$ and equation (9.5). The particular procedure employed would depend somewhat upon the knowledge of the geopotential existing at the time the determination process is initiated. A simplified scenario, based on assumed conditions that might exist, can illustrate the general character of the procedures.

First, it can be assumed that the coefficients are available for a reasonably accurate geopotential in spherical harmonics through 15, 15. Presumably this geopotential representation would be the product of orbital analyses of many satellites, perhaps including altimeter data as outlined in Section 3 and in Lundquist (1967a).

Second, it will be assumed that an altimeter-bearing satellite has been flown and has produced altitude data sufficient for a substantial improvement in the ocean geoid beyond the profile given by the 15, 15 solution. The accuracy of the altitudes should be in the few-meter range as discussed in Section 1.

Third, laser tracking of the satellite to meter accuracy will be assumed from a network of several ground stations.

Fourth, it will be assumed that the knowledge of the geopotential (including coefficients for resonant harmonics) together with ground-station tracking data allow an orbit for the satellite to be determined with meter accuracy in the vicinity of the tracking stations. However, the orbit based on ground-station tracking may not be expected to maintain meter accuracy over the long stretches of ocean between ground stations.

This last assumption leads to the fifth assumption, namely, that the altitude data themselves must be used to help generate the orbit to meter accuracy over the oceans. At the same time, the altitudes will be used to define the ocean geoid. An improvement of the geopotential probably will not be required for orbit determination, but if it is required, the determination can be done as a separate step in the chain of calculations.

Given these assumptions, the procedure to be followed is broadly a differential improvement calculation. The steps of the process might go as follows:

From the ground-station tracking, initial orbital elements and a corresponding initial orbit would be generated. From this orbit, a position $\vec{r}(t_a)$ of the satellite can be calculated for the time $t_a$ of each altitude measurement.

From the position of the satellite, and the ocean geoid corresponding to the initial geopotential, the point $\vec{S}$ on the ocean to which the altitude has been measured can be calculated. This calculation of course depends upon the characteristics of the altimeter. The calculation might be quite different for a broad-beam radio altimeter than for a narrow-beam laser altimeter. The calculated or expected value of the altitude is then the magnitude h of the vector

$$\vec{h}(t_a) = \vec{r}\,(t_a) - \mathcal{R}_3 \mathcal{R} \vec{S}[\vec{r}^{\circ}(t_a),\ V_0,\ L_1,\ L_2,\ \ldots\ L_M] \tag{9.8}$$

The position $\vec{r}\,(t_a)$ is to be considered as a function of the orbital elements $E_k$, although this dependency has not been explicitly indicated in equation (9.8). The transformations $\mathcal{R}$ and $\mathcal{R}_3$ carry the earth-fixed vector $\vec{S}$ into the space-fixed system in which $\vec{r}$ is expressed (see Gaposchkin, 1966, or Lundquist, 1967a).

Formally expressing the differential of h gives

$$dh = \sum_{i,k} \frac{\partial h}{\partial r_i} \frac{\partial r_i}{\partial E_k}\, dE_k + \sum_{\ell,i,k} \frac{\partial h}{\partial S_\ell} \frac{\partial S_\ell}{\partial r_i} \frac{\partial r_i}{\partial E_k}\, dE_k + \sum_{\ell} \frac{\partial h}{\partial S_\ell} \frac{\partial S_\ell}{\partial V_0}\, dV_0 + \sum_{\ell,j} \frac{\partial h}{\partial S_\ell} \frac{\partial S_\ell}{\partial L_j}\, dL_j + \sum \frac{\partial h}{\partial r_i} \frac{\partial r_i}{\partial L_j}\, dL_j \quad . \tag{9.9}$$

Because of the fifth assumption, it could be possible to neglect the last term, involving the dependence of h on the coefficients $L_j$ through the dependence of $r_i$ on the $L_j$.

If dh is identified with the difference between the calculated and observed altitudes, the equation (9.9) leads to an observation equation for corrections $\delta E_k$ to the orbital elements, and for corrections $\delta V_0$ and $\delta L_i$ to the parameters specifying the geoid. The observation equation derived from equation (9.9) must also provide for errors $\delta h$ in the observations of h. In conventional format, for each measurement $h_i$ at some time $t_i$ an observation equation is written (see Kaula, 1966, p. 72):

$$h_i(\text{observed at } t_i) - h_i(\text{calculated for } t_i) = \mathcal{M}_{ij}\delta F_j - \delta h_i \quad , \qquad (9.10)$$

where $\delta F_j$ are the corrections $\delta E_j$, $\delta V_0$, $\delta L_j$; and $\mathcal{M}_{ij}$ are the partial derivatives with respect to $F_j$ from equation (9.9) evaluated at time $t_i$.

The next step in utilizing altimetry data is a solution for the $\delta F_j$ from equations (9.10) and similar observation equations from ground-station tracking. The latter equations, by the assumptions above, will involve only the $\delta E_j$. Each of the equations (9.10) will have a term in $\delta V_0$. However, the occurrences of terms in $\delta L_j$ are strongly limited by the localized properties of the $\mathcal{J}_j$.

To illustrate this situation, a simplified procedure can be considered. The surface of the reference surface can be divided into $M = (N+1)^2$ areas centered on the points in the selected set from which the $\mathcal{J}_j$ were derived. For the region around point $(r_k, \delta_k, \lambda_k)$, the term $L_k \mathcal{J}_k$ dominates the sum in equation (9.7). Each line from the satellite through the subaltimeter point $\vec{S}_i$, for an observation at time $t_i$, intersects one of the M regions, noted for example by an index k, $\vec{S}_i^k$.

This suggests that a vector $\vec{T}_i^k$ be defined by

$$\vec{T}_i^k = \vec{S}_i^{k,0} - \vec{G}^{k,0} \quad , \qquad (9.11)$$

where $\vec{G}^{k,0}$ is the point with arguments $\theta_k, \lambda_k$ on the surface defined by (9.7) with the initial values $V_0^0$ and $L_j^0$ in the geopotential, and $\vec{S}_i^{k,0}$ is the subaltimeter point calculated with the initial $V_0^0$ and $L_j^0$. The vector $\vec{T}_i^k$ is a chord connecting two points on the geoid.

It is reasonable to expect that $\vec{T}_i^k$ will be essentially unaffected by small changes in $V_0$ and the $L_j$, since both terms on the right of (9.11) respond similarly to these changes. Thus $\vec{S}_i^k$ will be assumed to have a satisfactory representation as

$$\vec{S}_i^k = \vec{G}^k + \vec{T}_i^k \quad . \tag{9.12}$$

This asserts that the dependence of $\vec{S}_i^k$ on small changes in $V_0$ and the $L_j$ arises mainly through the dependence of $\vec{G}^k$ on $V_0$ and the $L_j$. But by the localized properties of the $\mathcal{F}_j$, only changes in the single coefficient $L_k$ contribute significantly to changes in $\vec{G}^k$. Hence,

$$\frac{\partial \vec{S}_i^k}{\partial L_j} = \delta_{jk} \frac{\partial \vec{G}^k}{\partial L_k} \quad . \tag{9.13}$$

The result of equation (9.13) is a substantial simplification of equations (9.10) through a simplification of the form of $\mathcal{M}_{ij}$. To illustrate this, consider the submatrix of $\mathcal{M}_{ij}$, which concerns corrections $\delta L_j$. This submatrix has M columns corresponding to the M independent coefficients $L_j$. There will be as many rows as there are altitude measurements.

The altitudes can be collected into M sets corresponding to the regions into which the corresponding $\vec{S}_i$ fall. The observation equations can be grouped in the same way, so that $\mathcal{M}_{ij}$ first has some number of rows corresponding to altitudes to the first region, then some other number of rows corresponding to the second region, etc.

Consider next the submatrix of $\mathcal{M}_{ij}$ corresponding to a region identified by index k. By equation (9.13) this submatrix has nonzero entries only in the k<u>th</u> column, where the entry in the row for the i<u>th</u> observation is

$$\left. \sum_{\ell} \frac{\partial h}{\partial S_\ell^k} \frac{\partial G_\ell^k}{\partial L_j} \right|_{t = t_i} \tag{9.14}$$

Note that here the subscript $\ell$ denotes the component of the vector, not the particular observation. The latter is explicitly indicated by the observation time at which the expression is evaluated.

The solution of the observation equations proceeds with formation of the normal equations. However, the form of $\mathcal{M}_{ij}$ in those columns that concern the $\delta L_j$ is just that for which the solution of the normal equations is greatly simplified. Kaula discusses this in detail, and his development will not be repeated here. (See Kaula, 1966, beginning on page 104. The matrix $\mathcal{M}$ has the form required in Kaula's equation 5.61 on page 105.)

In broad terms, when solving for the $\delta F_j$ there would be finally a largest submatrix to invert having the dimension of the number of orbital elements plus one for $V_0$. This is a vast simplification compared with inverting a matrix of this dimension plus M, particularly when M is very large.

As a final step, the $\delta F_j$ are added to their respective $F_j$, resulting in an improvement to both orbital elements and to the geopotential coefficients. If necessary the whole process can be iterated, beginning again with the improved elements and geopotential.

An additional feature of the procedure deserves mention. There will of course be no altitude observation equations for regions corresponding to land masses. Hence the coefficients $L_j$ for these regions will maintain their initial values, which presumably reflect the best information available from other arguments.

The scenario for a geopotential solution offered in this section has been drastically simplified and depends upon assumptions that may or may not prevail when an altimeter satellite is flown. In a more realistic treatment, most of the assumptions and approximations could be relaxed without essential effect on the general features of the procedures. Thus it seems likely use of functions $\mathcal{J}_j(r, \theta, \lambda)$ can significantly facilitate an improvement of the geopotential based on altitude observations. This allows the final conclusion that satellite-to-ocean altitudes promise substantial geopotential information in a form that can be analyzed without undue difficulty.

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ATTACHMENT D

# GUIDANCE THEORY AND TRAJECTORY ANALYSIS SEMINAR ABSTRACTS

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# REPRESENTATIONS FOR FINE GEOPOTENTIAL STRUCTURE

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The representation of a geopotential in spherical harmonics becomes increasingly awkward as requirements evolve for finer spatial detail. For example, treatments of satellite-to-ocean altitudes for 1-degree squares on the surface of the Earth would imply the summation of some 40,000 spherical harmonics to specify the geoid in each 1-degree square. A similar situation prevails in merging a satellite-determined geoid with detailed astrogeodetic geoids over continental areas.

Relief from this dilemma is possible by transformation to alternative sets of functions with the same dimension as a truncated set of spherical harmonics, but having the property that only a few terms in the alternative series contribute significantly to the potential at any point on the geoid.

The nature of such alternative functions is illustrated by the following transformation of the longitude dependence in a spherical harmonic expansion through indices n,n. Define $q_k(\ell)$ by

$$q_k(\ell) = \sum_{j=0}^{n} a_j \cos j(\ell - k \frac{2\pi}{2n+1}), \quad (k = 0,1,2,\ldots,2n)$$

where

$$a_o = \frac{1}{2n+1}, \; a_j = \frac{2}{2n+1} .$$

The inverse of these relations is

$$\cos h\ell = q_o + \sum_{j=1}^{2n} q_j \cos \frac{2\pi hj}{2n+1} \quad (h = 0,1,2,\ldots,n)$$

$$\sin h\ell = \sum_{j=1}^{2n} q_j \sin \frac{2\pi hj}{2n+1}.$$

The set of functions $\{ q_k(\ell), k = 0,1,2,\ldots,n \}$ is a set of orthogonal functions spanning the same linear manifold as the set $\{\cos h\ell, \sin h\ell, h = 0,1,\ldots,n\}$. At longitude $\ell_k = k \frac{2\pi}{2n+1}$, the function $q_k$ has the value 1 while all other $q_i$ have the value 0 and their sum remains small in the neighborhood.

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